Can People Learn about ‘Black Swans’?
Experimental Evidence

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How do people cope with tail risk? In a lab experiment that removed informational and incentive confounds, subjects overwhelmingly behaved like Bayesian learners. The results of simulations further revealed that if one is to survive under tail risk, one needs to follow the Bayesian approach, as all boundedly rational alternatives fail. These findings support the Bayesian assumption commonly made in prior studies on tail risk and model uncertainty, and they also demonstrate the importance of optimal learning under tail risk. (JEL C91, D83, D87, G02, G11)

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Ever since Mandelbrot (1957) and Fama (1965), an extensive literature has firmly established tail risk as an essential feature of financial markets.1 Yet little is known about how financial decision-makers cope with it, even though this question is of central importance both for modern finance theory and for investor decision-making. Finance researchers would like to better understand how investors learn under tail risk in order to put the modern rare disaster literature to the test. That literature resolves the equity, peso, inverse peso, and other canonical asset pricing puzzles by assuming that investors take into account the possibility of unlikely bad events (or “black swans”) that are not observed within the sample (Weitzman 2007). However, if in reality investors use adaptive expectations, whereby their decisions rely on only the

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1 See, for example, Gabaix et al. (2003, 2006) and Kelly and Jiang (2014).
The recent payoff history, one must worry that the rare disaster paradigm offers little insight into the aforementioned puzzles. At a more practical level, the 2007-2008 global financial crisis called into question the ability of markets to cope with tail risk and increased investor interest in protecting themselves against the occurrence of black swans. Real-world investors would like to know whether people have the cognitive capabilities to learn about tail risk optimally. They would also like to know if the adaptive learning approach, which they commonly use, is good enough in terms of economic performance when optimal (Bayesian) learning proves too difficult.

Even though the question of how individual investors learn about tail risk is of high interest for both finance research and practitioners, no prior studies have addressed this question. This study sets out to do so using experimental data from a controlled laboratory experiment, as this is the only approach capable of delivering an answer. Indeed, the competing hypotheses to be tested—(1) “Real-world investors learn about tail risk like intuitive Bayesian econometricians” versus (2) “real-world investors use adaptive learning in the face of tail risk (i.e., ‘sentiment’ prevails in a behavioral sense)—are not testable with field data because the data are consistent with both stories. Consider, for instance, how the financial system failed to anticipate the crisis of 2007-2008 (see Giles 2008). The immediate explanation, which is that people are not intelligent enough to forecast rare events (Hypothesis (2)), has been widely emphasized outside of science (see Taleb 2004). However, recent work shows that such forecasting failures are also compatible with Hypothesis (1), as they emerge in an equilibrium model in which the rational agent has incentives to neglect rare events and instead focus on forecasting everyday events (Dow and Bond 2017). As a second example, consider the empirical evidence that investor opinions about tail risk fluctuate substantially on a daily basis, as reflected in asset prices (e.g., Kelly and Jiang 2014; Gao et al. 2017). What drives those fluctuations? At first glance, one may think the data tell us that people’s beliefs are irrational (consistent with Hypothesis (2)). However, Ortlik and Veldkamp (2015) draw our attention to the fact that the data are also perfectly consistent with Hypothesis (1). They demonstrate that when the agents do not know the true underlying data-generating process (DGP hereafter) for the observed economic outcomes and instead estimate it like Bayesian econometricians, their beliefs fluctuate considerably. These fluctuations occur when new information is received from everyday data, and the agents use these data to update their beliefs about the probability of the occurrence of black swans. These two examples illustrate how we cannot use field data to infer how people adapt to tail risk.

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2 See Arthur et al. (1997) and Lo (2017), chapters 1 and 6, among others.
3 See, for example, Andrew Karolyi’s editorial at http://risfs.org/blog/tail-risk-is-not-crash-risk-or-is-it/.
4 See Malmendier and Nagel (2011), Greenwood and Shleifer (2014), and Barberis et al. (2015), among others.
They also illustrate why: two major confounding factors—the informational and incentive confounds—preclude inference. The incentive confound relates to the issue of being unable to control for real-world investor incentives; seemingly irrational behavior may in fact be a rational response to incentives that are unknown to the researcher. The informational confound relates to the issue of not knowing precisely what type of information about the structure of economy is available to the agents and what type is concealed from them. Since apparent updating failures may actually reflect structural uncertainty, the rational versus adaptive learning theories cannot be tested against one another using field data (Brav and Heaton 2002). By contrast, I was able to address both the informational and the incentive confounds by using controlled experimentation in a laboratory setting.

To capture tail risk in a laboratory setting, I designed a special decision-making task that involved learning whether a series of DGPs are normally distributed or fat-tailed. I derived theoretical predictions for learning the specific task based on Bayesian versus adaptive learning algorithms and compared them to the actual behavior of university students faced with the task in the lab. To remove the incentive confound and fully control for the subjects’ mindset at the time of the experiment, I provided them with significant incentives to perform the experimental task at a high level. The participants were provided with considerable monetary incentives, and I also ensured that the experimental task was sufficiently engaging so that the level of intrinsic motivation in the participants was high.

To remove the informational confound, I took three steps. First, I delivered the task instructions in an intuitive fashion to ensure that all of the task features were readily comprehensible by any task participant. Second, I removed any reference to financial concepts from the task. This was crucial given the documented extent of financial illiteracy among the general public. In a financial task, any factors associated with limited understanding of the task setting versus limited learning capabilities would masquerade for one another; this study sets out to isolate people’s learning capabilities regardless of their knowledge of basic finance concepts. Third, I described the stochastic structure of the task in exhaustive detail in the task instructions; that is, there was no structural uncertainty for the subjects. Injecting an element of structural uncertainty into the task would be problematic because subjects would inherently “fill in” missing information by imagining “their own scenarios.” As a result, the experimenter would lose control of the subjects’ prior beliefs about the task structure.

Since structural uncertainty was ruled out by design, and assuming that the subjects grasped the principles of the task (which I verified; see Section 2), the knowledge that the subjects possessed when they began the task amounted

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5 For example, Bernheim (1995) and Lusardi and Mitchell (2007).
to the information contained in the task instructions—no more and no less. Controlling for the subjects’ prior beliefs in this way, rather than making the prior beliefs a free parameter of the model, was an important aspect of constraining the Bayesian model used in the main analysis (see Section 1).

After running the lab experiment, I found that the large majority of the subjects acted like Bayesian learners in the experiment along three key dimensions, beginning with subject behavior. I designed the task in such a way that by observing the behavior of a given subject in each trial during the task, one can tell whether the subject acted like the Bayesian model in the experiment, as Bayesian learning in the task exhibits clear behavioral signatures (described in Section 1.2.3). The majority of the subjects behaved like Bayesians according to both an immediate “eyeball test” (subject behavior exhibits the aforementioned Bayesian signatures) and a more formal model comparison analysis of the Bayesian and adaptive models (the Bayesian model fits subject behavior markedly better). Second, going one step further in the analysis, I provided direct evidence that the beliefs held by the subjects throughout the task were Bayesian. To establish this, I elicited subjects’ beliefs regarding the type of DGP that they faced throughout the task. I found that the subjects’ beliefs closely matched those of the Bayesian model throughout the task. Third, subjects’ accumulated earnings at the end of the task were also Bayesian-like. The distribution of the earnings across subjects is akin to the distribution of Bayesian earnings (i.e., the earnings predicted by the Bayesian model when simulated in the exact same instances of the task as those that the subjects faced), whereas it is markedly different from the distribution of adaptive earnings.

The evidence also indicates that a majority of the subjects (although not all; see Section 3) acted as if they were Bayesians in all aspects of their decision-making, not just learning. The Bayesian model used in the core analysis not only predicts that the subjects implemented Bayesian learning to update their beliefs, but it also further predicts that they implemented the “model averaging approach” to make a decision in each trial on the basis of these beliefs. The model averaging approach—a core tenet of Bayesian statistics (Madigan and Raftery 1994)—consists of weighting all the possible world models (i.e., in the present setting, all the possible DGPs) based on evidence. In the prior finance literature on model uncertainty, it is common to assume that people engage in Bayesian model averaging. However, people may instead ignore model uncertainty and condition on the model deemed most likely, which is singled out as the best. That is, people may be Bayesians in their learning but “classical” in their decision rule. To explore this possibility, I compared the goodness-of-fit of the Bayesian model to that of a modified version of the model (henceforth, “the suboptimal Bayesian model”) in which the task participants

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place full weight on the DGP that they consider most likely; that is, from a Bayesian standpoint, the agent “jumps to conclusions.” The Bayesian model appears to fit subject behavior better than the suboptimal variant does for 90% of the subjects.

In simulations, I also found that applying Bayesian learning reaps substantial benefits in the face of tail risk. I compared the earnings of the Bayesian and adaptive learning models (according to a standard mean-variance criterion) in batches of 500 Monte Carlo simulations of the task. The Bayesian model markedly outperformed its competitor in those simulations, as well as in simulations in which the Bayesian agent held erroneous prior beliefs about the stochastic structure of the task, and in simulations in which outcome volatility was increased compared to that in the original task, in order to make the Bayesian estimates less reliable. This even held in simulations in which the aforementioned suboptimal Bayesian model was used in place of the benchmark model. In all of these simulations, the Bayesian earnings were not only significantly higher on average, but also less volatile than those obtained from applying the adaptive approach.

Finally, the evidence suggests that it is the presence of tail risk per se that provides an edge to Bayesian investors. In Monte Carlo simulations of a variant of the task in which tail risk was absent (the DGP was always normal), Bayesian and adaptive learners behaved alike. Furthermore, I found that the higher the level of tail risk, the larger the gap between Bayesian and adaptive earnings. To establish this, I ran batches of simulations of several variants of the task, each time with a different level of tail risk. For each tail risk level, the average gap between the economic performance of the Bayesian and adaptive models was computed. The gap appears to increase linearly with the level of tail risk in the environment.

My research question builds on both the long-standing literature demonstrating that tail risk is an essential aspect of financial markets and recent finance papers that rehabilitate the Bayesian learning hypothesis to understand how real-world investors cope with tail risk. On the theoretical side, Orlik and Veldkamp (2015) and Kozlowski et al. (2017) provide profound theoretical insights into how real-world investors may learn about tail risk in the same way as Bayesian econometricians do. Johannes et al. (2016) demonstrate the plausibility of the Bayesian hypothesis by showing that asset pricing data are consistent with the notion that investors simultaneously learn about parameters, hidden state variables, and even model specifications, like econometricians.

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7 For example, Mandelbrot (1957), Fama (1965), and Gabaix et al. (2003, 2006).
8 Also consistent with the Bayesian view, the existence of well-documented price features such as the “volatility skew” and so forth suggests that investors use Bayesian updating in response to major market events. See, for example, Bates (2000) and Farhi et al. (2014).
The current study also builds on well-established knowledge from neuroscience and psychology about how real people learn. It was important to discipline the behavioral models used in the model comparison analysis with solid neuroscientific data. A common pitfall of conducting this type of analysis is that if left undisciplined, the set of non-Bayesian models is potentially infinite, as there is always some “boundedly rational heuristic” (Gigerenzer et al. 1988) that can be tailored to fit behavior in a given environment. However, one must be concerned that these heuristics are devoid of neurobiological foundations and idiosyncratic to the experimental outcomes observed—as in reverse engineering (Newell 2005). To circumvent this issue, I ensured that the behavioral models proposed here have neurobiological underpinnings. Overwhelming neuroscientific evidence suggests that the human brain is geared to implement both the Bayesian and the adaptive approach, depending on the context; specific neural substrates have been identified for each approach (see Doll et al. 2012 for a survey).

This study contributes to several strands of literature in finance on both empirical and conceptual grounds. On a conceptual level, the current framework helps to make the otherwise-abstract notion of tail risk more tangible by highlighting what makes tail risk problematic from the standpoint of a financial decision-maker (see Section 1.2.1). The study also contributes to the finance literature on model uncertainty both empirically, by testing the Bayesian “model averaging” assumption commonly made in that literature (as explained above), and conceptually, by extending the notion of “model uncertainty” to DGP uncertainty—the uncertainty that emerges from not knowing whether asset returns are normally distributed or fat-tailed.9 Last but not least, the evidence for Bayesian learning documented here supports the aforementioned rare disaster hypothesis.

Finally, this study adds to the experimental literature on learning in economics and finance. It is well known that the Bayesian and adaptive learning approaches are “close cousins” (Camerer and Ho 1999) in the sense that they ultimately behave similarly in many domains. Here, I show that the presence of tail risk in the economic environment provides an edge to the Bayesian approach. Furthermore, the findings from my lab experiment contribute to the experimental literature on learning in financial markets:10 this is the first experimental study on tail risk in the literature.11

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9 The prior finance studies on model uncertainty provide key insights into portfolio choice by drawing the attention of the finance community to the importance of model uncertainty beyond parameter uncertainty. However, those studies operate in the familiar environment of linear, normal models, whereby “model uncertainty” reduces to uncertainty over which predictive variables to include among the candidate variables; that is, DGP uncertainty is ruled out.


11 In the work most similar to the current study, Payzan-LeNestour and Bossaerts (2015) provide evidence for Bayesian learning in a complex task involving the combined learning of jump and outcome probabilities. However, model uncertainty over tail risk was ruled out in that task.
1. Tail Risk in the Lab

1.1 Experimental design

1.1.1 Task description. The task features a bowman who, in each trial, is shooting at a target on a wall. The wall is represented by a line; the target corresponds to the zero mark on that line, and the shot realized at trial $t$, denoted $X_t$, can be anywhere on the line. In each trial $t$, the agent must decide whether to bet that the next shot will fall up to 4 meters away from the target on both sides (i.e., $X_t \in [-4; 4]$). A winning bet yields $2; a losing bet ($X_t \notin [-4; 4]$) results in a loss of $40. The alternative to betting is to “skip,” which yields $0 for certain. Immediately after deciding whether to bet or skip, the agent sees the realized shot and then proceeds to the next trial. See Figure 1. The goal is to maximize the outcomes accumulated from all trials.

One run of the task comprises 15 independent sessions of 20 trials each, and each session has a different bowman. There are two types of bowmen: “master” and “apprentice.” The shots from a master bowman are normally distributed around the target (i.e., the mean is 0) with a standard deviation that is unknown to the agent. The value of the standard deviation differs across master bowmen; it is uniformly distributed between 0.1 and 2. The shots from an apprentice bowman are Cauchy distributed around the target (i.e., the center is 0) with a dispersion of 1. At the beginning of each session, there is an equal chance of facing an apprentice or a master.

1.1.2 Information provided to the task participants. In each session, the task participant is not told whether the bowman is a master or an apprentice; additionally, the value of the standard deviation of the shots from master bowmen is a random and hidden parameter. Thus, like real-world investors, task participants face model uncertainty over tail risk, beyond parameter uncertainty: not only do the task participants not know key parameter values within a given distributional form but also the nature of the distributional form itself is unknown. However, there is no structural uncertainty for the agent. That is, the task participants know the structure of the task (15 independent sessions, each comprising 20 trials, so 300 trials altogether) and that, in each session, there is an equal chance of facing an apprentice or a master. The participants also know that the shots from a master are normally distributed with zero mean and a standard deviation uniformly distributed between 0.1 and 2, whereas the shots from an apprentice are Cauchy distributed with a center of 0 and a dispersion of 1.

As explained in the Introduction, the absence of structural uncertainty in the laboratory experiment helps to eliminate the informational confound by controlling for the subjects’ prior beliefs. If the subjects’ prior beliefs were a free parameter that could therefore be fine-tuned to fit actual behavior, one would be concerned that the Bayesian learning model may fit actual behavior better than the adaptive model does due to overfitting. To circumvent this issue,
I disciplined the Bayesian model used in the model comparison analysis: the model has no free parameters, except those related to risk attitude.

For the same reason (removing the informational confound), all of the information contained in the task instructions about the stochastic structure of the task was provided to the task participants in a readily accessible format. Any technical terms from statistics and finance were avoided. Furthermore, the task instructions, rather than stating the expected value of betting under each type of Bowman, allowed the subjects to actually experience those values using the method of Gigerenzer et al. (1988). Specifically, the subjects viewed an animation or “distribution builder” showing 300 successive sample shots from an apprentice and three other distribution builders showing 300 successive shots from different master bowmen—the first with the minimum level of standard deviation (0.1), the second with the maximum level (2), and the third with an intermediate level (1). These animations can be viewed at
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http://bowmanexperiment.weebly.com/task-instructions.html.¹² Note that the use of the bowman narrative reflects the same general strategy of facilitating information acquisition in the subjects by avoiding the use of technical (in particular, financial) concepts. As explained in the Introduction, this aspect of the method was key in order to be able to isolate the nature of learning in the subjects.¹³ The experimental results validate this strategy by suggesting that the subjects grasped the main features of the task (see Section 2).

1.1.3 Intuition for the current task specification. The task is best understood as a stylized representation of the credit market. Apprentice bowmen represent assets that combine high levels of tail risk and low payoffs, as did many asset classes in several periods of financial history. Examples include high-yield bonds in 1989 (e.g., Altman 1987, 1992) and RMBS CDOs in 2007 (e.g., Crouhy et al. 2008). Master bowmen represent investment-grade bonds or government bonds.

My main motivation for using the Cauchy distribution to model fat-tailed payoffs is tractability. Using other fat-tailed distributions (the Levy distribution in particular; see Mandelbrot 1960) would not change the key features of the theory proposed in Section 1.2, but it would make it more complicated. The motivation for using this particular outcome structure (the feature of payoff discontinuity around the bounds -4 and +4 in particular) and the aforementioned parameter values for payoff levels, standard deviations, dispersion parameter, etc., was that the Bayesian and adaptive models behave differently under this specification of the task, as I show next. One can therefore neatly distinguish between Bayesian and non-Bayesian behaviors. I tested other parameter specifications in Monte Carlo simulations; the current specification appeared to exhibit the best statistical power for the model comparison analysis.¹⁴

¹² This aspect of the experimental design is grounded in the distinction between experienced and stated representations of randomness. Psychologically, important differences in the perception of randomness are stated in the form of summary statistics (such as reading the probability 0.25 or reading that the distribution is normally distributed with a mean of 0 and a standard deviation of 1) and randomness that is experienced (such as seeing 25 of 100 coin flips turn up heads or seeing 300 samples drawn from the standard normal distribution); see, for example, Gigerenzer et al. (1988) and Goldstein et al. (2006).

¹³ Note that the absence of any analytical notion in the task instructions was also important to exclude the possibility that I induced (“primed”) my subjects to implement Bayesian learning. Some would indeed argue that if present in the task instructions, analytical concepts would potentially work as cues about what constitutes expected behavior in the experiment. That is, by framing the task as an analytical problem, one would induce the subjects to enter into some sophisticated mode of thinking (Evans 2003).

¹⁴ Among other choices, I ran Monte Carlo simulations of versions of the task with different parameter values for both gain and loss outcomes. I also varied the value of the bounds of the standard deviation parameter for the normal DGP and considered a version of the task in which there is only one possible value for the standard deviation; that is, the agent faces no uncertainty over the standard deviation of the normal DGP (I tested different values for the standard deviation). None of those versions improved on the current one in terms of maximizing discriminatory power in the model comparison analysis. Note that I am not claiming that the current design is optimal, but the results show that it was sufficient for my purpose here.
1.2 Behavioral models: Bayesian versus adaptive

A formal presentation of the models can be found in Appendix B. The hallmark of the Bayesian approach is to assess the nature of the DGP in each session of the task. In principle, expected value is computed through “model averaging”; that is, by weighting the two possible scenarios—Hypothesis 1: the DGP is normal (bowman is a master) versus Hypothesis 2: the DGP is fat-tailed (bowman is an apprentice)—based on evidence. To account for the possibility that people cannot hold several hypotheses in their minds at one time, I also considered the suboptimal Bayesian model which uses a classical or “inductive” (Arthur 1994) decision rule instead of model averaging; that is, full weight is placed on the scenario deemed most likely. Whatever the version of the model, Bayesian behavior consists of betting if the evidence that the DGP is normal is above a threshold whose specific value depends on the agent’s attitude toward uncertainty—the more loss averse or the more disappointment averse the agent is, the larger the threshold.

If implementing the Bayesian approach proves excessively difficult for the task participants, the alternative approach, adaptive learning, can take many forms in the current task. The generic “experience-weighted attraction” (EWA) learning approach (e.g., Camerer and Ho 1999; Pouget 2007) encompasses a wide spectrum of adaptive approaches, from pure reinforcement learning (which focuses on actually experienced outcomes only, ignoring counterfactual outcomes altogether) to “weighted fictitious play” (which equally weights both types of outcomes). The common denominator among all EWA models is that they apply the “law of effect” (Thorndike 1898), or the behaviorist principle according to which successful actions tend to be repeated. By automatically deeming an action good if it yielded good outcomes in the past, the EWA approach obviates the need to infer payoff probability. In contrast, the “adaptive forecaster” model estimates payoff probability using the adaptive forecasting algorithm of Foster and Vohra (1998). Starting with an estimated probability of a winning bet that reflects the true prior probability, the adaptive forecaster model updates his probability estimate on each trial according to the frequency of occurrence of shots within the winning range (\([-4, 4]\)) until and including the current trial. Observing a low frequency of occurrence leads the agent to decrease his probability estimate. Although the adaptive forecaster model may seem to resemble the Bayesian model on the surface (both aim to estimate payoff probability), the two approaches are fundamentally different across two key dimensions: in contrast to Bayesians, adaptive forecasters are both backward-looking (they do not learn anything about the future from past data) and completely ignorant of the reasons that winning bets occur or fail to occur (the DGP is completely ignored).

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15 The prior probability of a winning bet is approximately \(0.5 \times 0.99 + 0.5 \times 0.84 = 0.91\) because by design there is an equal chance of facing a master and an apprentice, and the probability of a winning bet is approximately 0.99 on average with a master and 0.84 with an apprentice.
There are two main distinctive features of Bayesian behavior in the present task. First, the Bayesian agent systematically skips in the first trials of each session. Second, he does not systematically skip after observing a shot outside the winning range. Instead, the agent persists in betting after a shot that fell sufficiently close to the target (and thus, the evidence that the Bowman is a master remains high). Adaptive agents tend to do the opposite by betting earlier than Bayesians do in each session and by systematically skipping after a loss outcome, regardless of the value of the shot that led to the loss. This is because, by nature, the Bayesian agent chooses to bet only when/if the evidence that the Bowman is a master is sufficiently large. Adaptive agents ignore such evidence and focus solely on past realized outcomes. Thus, unless a bad outcome is experienced in the first trials of the session, betting is attractive to the EWA agent in early trials. The same is true for the adaptive forecaster, but not for the same reason: the agent begins with an estimated probability of a winning bet that is quite high (close to the true prior probability of a winning bet, as explained above), and the occurrence of shots within the winning range confirms the initial guess. Similarly, the occurrence of a single shot outside the winning range is sufficient to deter both adaptive agents from betting, either by making action bet appear unattractive (for EWA) or by causing the estimated probability of a winning bet to decrease (for the adaptive forecaster).

To better appreciate the ways in which the models behave differently, consider Figures 2 and 3, which show how the three models behaved in one
Figure 3
Learning about normally distributed payoffs using the adaptive models. In this simulated run of a session with a master Bowman, the EWA model (top) and adaptive forecaster model (bottom) faced the same data as the Bayesian model in the simulation reported in Figure 2. The top graph reports the value of action skip ($A_S$) and the value of action bet ($A_B$) for the EWA model and the estimated probability of a winning bet ($P$) for the adaptive forecaster model. See Figure 2 for the legend of the bottom and middle graphs.
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simulated run with a master bowman. Figure 2 shows that the Bayesian model skipped in the first trials of the session, when the evidence for the normal DGP (“$B_{12}$”) was still low due to the insufficient sample size. The model successfully learned that the bowman was a master and hence consistently bet from trial 8 onward. Note how the model persisted to employ action bet despite losing in trial 16, as the data from trial 16 onward continued to favor the hypothesis that the bowman was a master (in the top graph, note how $B_{12}$, the evidence that the bowman was a master, was above the dotted line, which defines the threshold value of $B_{12}$ from which the Bayesian model chooses to bet). Figure 3 shows that the EWA model began to bet in the session in trial 5 and chose to skip after the bad outcome occurred in trial 16, as the occurrence of that outcome depressed the value of action bet ($A_B$ in the top graph for the EWA model).\textsuperscript{16}

The adaptive forecaster model behaved essentially the same as the EWA model, but not for the same reason: model choice in that case reflects the evolution of the probability of a winning bet estimated by the model ($P$ in the top graph for the adaptive forecaster); the estimate was high at the beginning of the session (thus, the model chose to bet) and low after the shot outside the winning range occurred at trial 16 (thus, the model chose to skip).

This example helps clarify why setting payoff discontinuity around the bounds $[-4, 4]$ was a pivotal design feature of my study. Without such a discontinuity, the EWA model would not systematically skip after experiencing a loss. To see why, consider the alternative continuous setting in which the loss amount corresponds to the value of the shot that led to the loss (e.g., a shot of 4.5 leads to a loss of $4.5$). In such a setting, the value of action bet is not sufficiently reduced; hence the EWA agent continues to bet after a near miss occurs. That is, the key Bayesian signature that consists of betting after a near miss in the sessions with a master is lost.

Note that the EWA model bet early in the simulated session reported in Figure 3, despite the fact that I set the initial value of action skip to a quite high level to make the model skip in the first trials of the session; for smaller initial values, the model would begin betting even earlier.\textsuperscript{17} The key point here is that by fine-tuning the (many) parameters of the EWA model, it is possible to partially mimic one of the two Bayesian signatures, but not both simultaneously (see Appendix B.2 for details).

Figures 4 and 5 show how the three models behaved in one simulated run with an apprentice bowman. The Bayesian model successfully learned that the bowman was an apprentice (as reflected by the very negative value of $B_{12}$).\textsuperscript{18}

\textsuperscript{16} In those simulations, the models use the optimal (deterministic) choice rule in which the action with the higher value is chosen with probability 1. Note that using a stochastic choice rule such as the logit rule instead leads the EWA model to sometimes bet after losing as a result of “trembling” if the randomness coefficient of the logit rule is sufficiently large. But such a model is not a good fit to subject behavior, because subject behavior was not random in the present experiment; more on this in the Results section.

\textsuperscript{17} In theory, the gap between the initial values of the two actions should be 1.57; that is, $(A_B(0), A_S(0)) = (-1.57, 0)$ (see Appendix B.2). The gap was 4 in this simulation.
Learning about heavy-tailed payoffs using the Bayesian model.

In this simulated run of a session with an apprentice bowman, black-swan realizations (shots well outside the winning range) occurred in both trial 10 and trial 15. \( X_{13} \) approached 4.5. The other shots fell within the winning range. See Figure 2 for the legend of each graph.

1.2.1 Why is tail risk a plague for both types of learners?.

The foregoing highlights the main reason that tail risk is a plague for adaptive learners: after observing a series of good outcomes, the agents are lured into betting. The key point here is that winning streaks regularly occur under the fat-tailed DGP; hence, the adaptive agent often bets, thereby exposing himself to large losses (as illustrated in Figure 5). Although forecasting black swans allows the Bayesian
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Figure 5
Learning about heavy-tailed payoffs using the adaptive models.
In this simulated run, the EWA model (top figure) and adaptive forecaster model (bottom figure) faced the same data used in the Bayesian model in the simulation reported in Figure 4. See Figure 3 for the legend of each graph.

model to protect itself against this type of loss, it is important to realize that tail risk is an irremediable plague even for the Bayesian model, as forecasting black swans is not always possible. For an example, consider Figure 6, which shows how the Bayesian model was deluded into believing that the DGP was Gaussian in one simulated run of a session with an apprentice bowman. In that simulation,
In this simulation of the Bayesian model in a session with an apprentice, all the simulated shots were Gaussian-like (they all fell up to 3 meters away from the target) until the last trial, when a black-swan realization eventually occurred ($X_{20} \approx 19$). See Figure 2 for the legend of each graph.

Figure 6
Tail risk and camouflage.

In this simulation of the Bayesian model in a session with an apprentice, all the simulated shots were Gaussian-like (they all fell up to 3 meters away from the target) until the last trial, when a black-swan realization eventually occurred ($X_{20} \approx 19$). See Figure 2 for the legend of each graph.

the data were Gaussian-like (i.e., the shots fell well within the winning range) until a black swan eventually occurred in the last trial of the session, revealing that the DGP was in fact Cauchy. Metaphorically, it appears that the Cauchy DGP is good at “camouflaging” itself. Those instances of camouflage are the root cause of the Bayesian model sometimes losing money under tail risk. In our example, the evidence that the bowman was a master ($B_{12}$ in the top graph) became large enough so that the Bayesian model chose to bet from trial 9. The model therefore lost $40 on the last trial when the black swan took the agent by surprise.

Those instances of camouflage are not an exception but the norm under tail risk. Seemingly normal realizations regularly occur. By tricking the Bayesian and luring the adaptive models, those realizations irremediably render both types of decision-makers vulnerable to tail risk. To further appreciate the hallmark of tail risk, consider Figure 7, which reports the results of one simulated run with the EWA model in a “special” session with a master bowman. In that session, the normal DGP had mean $-2$ and standard deviation 3, making the environment maximally hostile for the decision-maker (recall that the mean was 0 and the standard deviation was at worst 2 in the main task). Such a setting represents the most dangerous environment that one can imagine for the agent within the set of normal worlds, inasmuch as bad outcomes (should the agent bet) occur very often, and volatility is very high. Yet it is not dangerous compared to the case in which the decision-maker faces tail risk. This is because even the most unsophisticated type of adaptive agent typically does not lose any money in the dangerous-yet-normal world, as bad outcomes regularly occur, and
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Figure 7
Decision-making when payoffs are normally distributed with negative mean and high volatility.
In this simulation of the EWA model with a master bowman, the DGP was modified to mimic normally distributed payoffs that are both extremely bad on average (the mean of the DGP was set to $-2$) and extremely volatile (the standard deviation was 3). See Figure 3 for the legend of each graph.

the agent is therefore wary of bad outcomes. For instance, in the simulation reported in Figure 7, the EWA model recognized that betting would have led to a loss on trial 3, and this protected the model from betting thereafter. Thus, the model did not lose any money, contrary to that under tail risk. As such, the “worst-case scenario” for the normal DGP makes an interesting counterpoint to the case in which the decision-maker faces tail risk by underscoring what makes tail risk unique for the decision-maker.

2. Results

2.1 Evidence for Bayesian learning in the lab

2.1.1 Experimental procedure. The participants in the lab experiment were undergraduate students (N=124, 64% male) from the University of New South Wales. The participants were directed to watch the set of online instructions for the task before the experimental session. These instructions informed the participants that they would be performing a demanding decision task, described the task, and emphasized that all of the accumulated outcomes from the task would be added to a starting account balance to determine each

18 A fortiori, the Bayesian model never bets in such a world because it learns very quickly that the expected value of betting is negative.
participant’s payoff up to $110. The participants were also told that the account balance could be anywhere between −$500 and $500 and that it would be revealed to the participants after the experimental session. Ample examples were provided to ensure that all the subjects understood how the account balance worked. Thus, by design, the subjects did not know the current value of their wealth (their net accumulated outcomes + the amount of the account balance) during each trial. The motivation for using the account balance feature was to prevent potential wealth effects from occurring during the experiment by making their current wealth level unknown to the subjects until after the task had been completed.

I set the amount of the account balance before the subjects began the task. I wrote the amount on a sheet of paper and placed the sheet in an envelope in the middle of the lab room; all this was done in front of the subjects. The amount of the account balance was set such that the average payoff in the session would not exceed $70 (the maximum cap achievable given budgetary constraints) based on forecasted payoffs. To forecast the payoffs, I used the payoffs from the previous sessions or, for the very first sessions, average simulated Bayesian earnings from the Monte Carlo simulations of the task. Payoffs ranged from $5 (the “show-up reward,” which was given independent of subject performance as per the lab protocol) to $110 (the cap set by the payoff rule), with a mean of $63 (the standard deviation was 42). Forty subjects ultimately obtained the minimum payoff (i.e., $5), and fifty-nine subjects received more than $100. Thus, the task participants were provided with considerable monetary incentives that were larger than those in similar experiments (except in the study by Payzan-LeNestour and Bossaerts 2015, which used a similar payment procedure), and were meaningful relative to their standard of living. As explained in the Introduction, providing large incentives to the subjects helps to eliminate the incentive confound, which is a general concern for any empirical study and a particularly pressing one here given the complexity of the current task.

Upon arrival at the lab, the participants again watched the online instructions for 30 minutes, after which they completed a multiple-choice questionnaire that confirmed their understanding of the task. After 15 minutes, the experimenter reviewed the answers with the participants. The participants were also briefed again on the stochastic structure of the task and the payment procedure. Subsequently, the participants completed one run of the task, which lasted for approximately 30 minutes.

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19 For example, if the accumulated outcomes from the task are $120 and the account balance has been set to −$100, the subject ends up with $20. If the accumulated outcomes are −$80 and the account balance has been set to $110, the subject ends up with $30, etc.

20 This procedure was important to ensure that the experimenter was not suspected of “cheating” (changing the amount after observing the subjects’ performance).

21 See the ample evidence that participants in complex cognitive tasks do not behave in a sophisticated way unless they are given high monetary incentives. For example, Wilcox (1993), Herwig and Ortmann (2001), and Charness et al. (2010).
2.1.2 Estimation procedure. I fitted the Bayesian and adaptive models to each task participant’s choices and then compared the goodness-of-fit of the best-fit Bayesian and best-fit adaptive models. A cross-validation procedure was used to fit the Bayesian and EWA models. Essentially, the procedure proceeded in two steps, a first “in-sample” calibration (estimation) step followed by an “out-of-sample” validation test. In the first step, the free parameters of each model were chosen separately for each subject to maximize the proximity of the model predictions to the actual choices. The parameters were optimized by minimizing the total squared prediction error compounded over the set of trials for the first eight sessions. The ensuing parameter estimates were then used in a second stage of the analysis to predict the choices in the last seven sessions (out-of-sample validation). The success rate of each model was measured by the percentage of trials in which the model successfully predicted the subject’s choice in the last seven sessions. For each subject, the out-of-sample success rate of the best-fit EWA model was compared to that of the best-fit Bayesian model—loss averse or disappointment averse—and, for reference, to that of the risk-neutral version of the Bayesian model for which there was no free parameter (i.e., no in-sample calibration was needed). Appendix C provides a more detailed description of the optimization procedure.

2.1.3 Main results.

2.1.3.1 Choices. The majority of the subjects acted as Bayesians. This finding was immediately apparent from the basic descriptive statistics, which revealed that subject behavior exhibited the aforementioned Bayesian features. That is, the subjects systematically skipped in the early stage of each session, and they consistently bet when there was sufficient evidence that a session was with a master Bowman, even when some realizations fell outside of the winning range. Specifically, 93% of the subjects skipped, at a minimum, the three first trials in every session of the task. The mode of the distribution of the first trial when the subjects bet (in those sessions in which they did eventually bet) is 6; see Figure 8. Furthermore, 76% of the subjects bet in sessions with a master Bowman after encountering a shot that fell between 4 and 7 meters away from the target.

The conclusions drawn from the formal model comparison further support the idea that subjects acted like Bayesians in the task. First, I found that the Bayesian model fits subject behavior unambiguously better than the suboptimal Bayesian model for 90% of the subjects; the p-value of a paired t-test based on the differences in the individual fits is null. As explained above, this finding supports the notion that subjects were Bayesians not just in their learning but also in their decision-making itself. Furthermore, the Bayesian model fits markedly better than the best-fit EWA model. A t-test leads to the rejection of the null hypothesis that the fits of the Bayesian and best-fit EWA models are
Figure 8
Distribution of the first time that the subjects bet in the task.
The distribution was derived across all subjects and sessions. N, cases in which the subjects skipped in all trials of a session.

Figure 9
Comparative goodness-of-fit of the Bayesian and EWA models.
The success rate (fraction of trials in which the model predicted the choice made by the subject) of the best-fit Bayesian model (x-axis). The success rate of the best-fit EWA model. Each data point corresponds to one subject (N=124) (y-axis). The Bayesian model fits better when the data point is below the 45 degree line. Equal (p-value is null). Figure 9 indicates good fits for the Bayesian model, with prediction success above 85% for a majority of the subjects. Notably, despite the fact that assuming risk neutrality is an oversimplification of actual behavior, the risk-neutral Bayesian model also fits subject behavior significantly better than the best-fit adaptive model for the majority of the subjects (p-value of paired t-test: .007). Over all subjects and trials, the average success rate is approximately 87% for the Bayesian model and 77% for the EWA model. It is 74% for the best-fit adaptive forecaster model. The goodness-of-fit of the EWA...
model is better than that of the adaptive forecaster model ($p$-value of paired t-test: .02).

**2.1.3.2 Beliefs.** The foregoing thus suggests that the subjects behaved like Bayesians in the experiment. Going one step further, I sought to directly elicit the beliefs of the task participants during the task in addition to inferring the nature of those beliefs from the subjects’ choices. For that purpose I ran additional experimental sessions that replicated the conditions of the original experiment except that at the end of each session, the subjects (N=52; same cohort as in the original experimental sessions: undergraduates at the University of New South Wales) were also asked whether they believed that the bowman in the session was a master or an apprentice. An important aspect of this experimental treatment was that a correct answer yielded $10 and an incorrect answer resulted in a loss of $10. The absence of a reply led to a loss of $12. This rule was well emphasized both in the task instructions themselves and in the multiple-choice questionnaire at the end of the task instructions to ensure that the subjects were provided with the proper incentives.

I found that the subjects correctly guessed the nature of the bowman in 84% of the sessions. Thirty-one subjects correctly assessed the type of bowman in all but one or two sessions at most. Eight subjects managed to assess the bowman’s type in every session. Strikingly, 42 subjects answered exactly like the Bayesian model in all but one or two sessions at most, and 23 subjects replied like the Bayesian model in all sessions. Like the Bayesian model, those subjects correctly identified the type of bowman in all the sessions except in instances when the Cauchy DGP was camouflaged by a series of Gaussian-like realizations (see figure 6, p. 4830). Taken as a whole, the subjects responded like Bayesians in the vast majority (92%) of the sessions. These results strengthen the evidence of Bayesian learning in the original experiment inasmuch as they rely on direct elicitation of the subjects’ beliefs.

**2.1.3.3 Earnings.** The evidence that subject behavior is well described by the Bayesian model can be further strengthened by focusing on the subjects’ net accumulated outcomes at the end of the task (henceforth, “subject earnings”). Average subject earnings were $81 (standard deviation: 94), which is below the best-fit Bayesian model average of $112 (obtained in simulations that replicated the conditions the subjects faced) and above the best-fit EWA model average of $51. However, when I excluded from the analysis a group of subjects whose behavior should be analyzed separately (see Section 3), I found that subject earnings closely resemble those of the Bayesian model: they average $120 (standard deviation: 58) versus $135 for the Bayesian model (standard deviation: 68). I could not reject the null hypothesis that the two means are equal ($p$-value of t-test: .08). Furthermore, I am able to reject the hypothesis that the distribution of the earnings across subjects and the corresponding
distribution of the simulated Bayesian earnings (see Figure 10) are alike in a Kolmogorov-Smirnov test (even at a threshold of 5%). In contrast, a similar Kolmogorov-Smirnov test unambiguously leads to the rejection of the hypothesis that the earnings of the best-fit adaptive model are distributed like the subjects’ earnings ($p$-value is close to 0).

### 2.2 Superior economic performance of the Bayesian approach under tail risk

#### 2.2.1 Comparison of Bayesian and adaptive earnings in the task.

I computed the distribution of the earnings from applying each model (Bayesian, EWA, and Adaptive Forecaster) across 500 simulated runs of the task and compared the earning performance of the models according to a standard mean-variance criterion. Figure 11 shows that the earnings of the Bayesian model average 140 (SEM: 3.6) versus 46 (SEM: 6.2) for the adaptive forecaster and 52.2 (SEM: 6) for the EWA model. The average economic performance of the Bayesian model is thus markedly better than the performance of each adaptive model (the $p$-value of a Welch $t$-test is essentially null). The economic performance of the Bayesian model is also superior along the variance dimension (the variance is smaller). Thus, the Bayesian model appears to make more money and to be less risky than the other two models.

#### 2.2.1.1 Robustness checks.

The foregoing simulation results were obtained under the assumption of risk neutrality, but they are robust to different risk
attitude assumptions. The prospect theory Bayesian model earned on average 131.9 (SEM: 3.67); the disappointment-averse variant of the model earned 124 (SEM: 2.79). The superiority of the Bayesian model to the adaptive models is also robust to different environmental conditions. Notably, in simulations of a volatile version of the current task in which the upper bound of the standard deviation of the shots coming from a master bowman was 3 (vs. 2 in the original task), the superiority of the Bayesian model was markedly enhanced: the earnings from applying Bayesian learning average 64.2 (SEM: 3.6), those from applying Adaptive Forecasting average −60 (SEM: 5.2), and those from the top-performing EWA model average −39.7 (SEM: 4.9). See Figure 12.

In light of the result of Foster and Vohra (1998) that the adaptive forecaster model is well calibrated in the long run, one may wonder whether prolonging the duration of each session equalizes the economic performances of the models. The current simulated data suggest that this is not the case; that is, a longer duration does not equalize the performances. In simulated runs of the task in which each session comprised 200 trials (instead of 20 in the original task), the economic performance of the adaptive forecaster model was similar to the performance of the top-performing EWA model. However, the Bayesian model still trumps both contenders (p-value of Welch t-test: 0.016). Therefore, under tail risk, the Bayesian model is consistently better across all environmental conditions, including in stable environments in which the sessions last ten times longer than in the original task.

The Bayesian earnings also appear to be barely affected when the Bayesian agent holds erroneous prior beliefs. I ran simulations in which the Bayesian model expected the environment to be less volatile than it actually was (the
The box-and-whisker plots represent the distributions of the earnings of the models—Bayesian (left), adaptive forecaster (middle), and EWA (right)—across 500 simulated runs of the task. In these simulations, volatility was higher than in the main task used in this study: the maximum level of the standard deviation in the sessions with a master bowman was 3 (it is 2 in the main task). The boxes represent the interquartile range (25th to 75th percentile); see the legend of Figure 11. Because the notch in the box plot of the Bayesian model does not overlap with the notch in the box plot of either the EWA or adaptive forecaster model, I conclude, with 95% confidence, that the median earnings of the Bayesian and adaptive models differ.

Table 1

Bayesian earnings in simulations in which the Bayesian model held erroneous prior beliefs about tail risk. The table reports the average earnings (SEM in parentheses) from applying the Bayesian approach in batches of simulations in which the Bayesian model held erroneous prior beliefs about the probability of facing a master bowman (the true probability was 0.5—top line, for reference).

<table>
<thead>
<tr>
<th>Prior probability that bowman is a master</th>
<th>Bayesian earnings</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>140 (3.6)</td>
</tr>
<tr>
<td>0.6</td>
<td>128 (3.7)</td>
</tr>
<tr>
<td>0.8</td>
<td>115 (4.2)</td>
</tr>
<tr>
<td>0.4</td>
<td>133 (3.2)</td>
</tr>
<tr>
<td>0.2</td>
<td>121 (2.6)</td>
</tr>
</tbody>
</table>

model assumed that the upper bound of the standard deviation of the shots from a master bowman was 1, whereas the actual level was 2). The average earnings from applying Bayesian learning are 110.5 in this case (SEM: 3.8). Similarly, simulations were run in which the Bayesian model considered the environment to be more volatile than it actually was (the model assumed that the upper bound of the standard deviation was 3). The average earnings are 125 in this case (SEM: 3.5). Finally, I ran batches of simulations in which the Bayesian model held erroneous prior beliefs about tail risk. Even in that case, applying the Bayesian approach reaps significant benefits; see Table 1.

Finally, I found that the superiority of Bayesian learning is also robust to simulating the suboptimal Bayesian model in place of the Bayesian model. In a batch of 500 simulations, the suboptimal Bayesian model earned on average 115.1 (SEM: 4.04). Its economic performance is thus superior to that of the
top-performing EWA model according to a Welch $t$-test ($p$-value: .0001). Thus, tail risk appears to consistently provide an edge to Bayesian learners, even those learners that jump to conclusions.

### 2.2.2 Evidence that the presence of tail risk is what provides an edge to Bayesian learners.

To investigate the precise role of tail risk in the superiority of the Bayesian model, I assessed the earnings of each model in a batch of 500 simulations of a tail-risk-free variant of the task. In that variant, all the sessions were with a master Bowman; thus, tail risk was absent. Figure 13 documents that the Bayesian model did not outperform the adaptive models in these simulations. This result was expected in light of prior work demonstrating the equivalence of Bayesian and adaptive learning when the payoffs are normally distributed (Aoki 1987; Camerer and Ho 1999). Together with the finding reported in Figure 11, this finding suggests that the outperformance of the Bayesian approach is derived from the presence of tail risk per se.

The evidence that tail risk is what provides an edge to the Bayesian model is further strengthened by Figure 14, which shows that the higher the level of tail risk in the environment is, the larger the gap between Bayesian and adaptive earnings. To establish this, I ran a batch of 500 simulations of the task for each of the following probabilities of being in a session with an apprentice: 0.5 (as in the main task), 0.6, 0.7, 0.8, 0.9, and 1. For each probability, the average gap between the economic performances of the Bayesian and adaptive models was computed. The gap appears to increase linearly with the level of tail risk used in the simulations.
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Tail risk level

0.5 0.6 0.7 0.8 0.9 1

Incremental earnings from applying Bayesian learning rather than EWA

Figure 14
Average gap between Bayesian and adaptive earnings for different levels of tail risk in the environment.

Level of tail risk in the task, measured by the probability that the shots come from an apprentice bowman (i.e., DGP is heavy-tailed) (x-axis). In the original task, the probability was 0.5. For each probability level, 500 simulations of the task were run with the Bayesian and EWA models, and the gap between the Bayesian and EWA models’ final earnings was recorded. The average earnings gap across simulations is reported, as well as the SEM (bar) (y-axis).

3. Discussion

The evidence for Bayesian learning documented in this study provides support for the Bayesian assumption commonly made in prior finance studies on tail risk and model uncertainty. More generally, the evidence supports the Bayesian explanations that have swept through cognitive science over the past two decades, from intuitive physics and causal learning to perception and language.22 Yet people flounder with even the most elementary probability questions.23 What explains this apparent paradox? How can a supposedly Bayesian brain fare so poorly with probabilities? The answer proposed by modern psychologists is that for the brain, explicitly representing probability distributions is difficult but drawing samples from that distribution is quite natural, inasmuch as the brain is a “Bayesian sampler” (Sanborn and Chater 2016). That is, the brain does not calculate probabilities (and is not adapted to do so) but instead draws samples from probability distributions. That would explain why people fare well in complex learning problems that involve sampling, such as the current task, but struggle to master even the most elementary explicit probabilistic reasoning.

The foregoing note highlights one key aspect of the protocol used in the present experiment, namely, the method through which the subjects were acquainted with the decision-making task in the task instructions. As explained in Section 1.1.2, to help the subjects grasp the statistics underlying the task, I allowed them to directly sample many times from the different DGPs that

22 For example, Wolpert (2007), Kemp and Tenenbaum (2009), Sanborn et al. (2013), Pantelis et al. (2014), and Petzschner et al. (2015).

23 For example, Tversky and Kahneman (1973, 1974), Charness and Levin (2005), and Elqayam and Evans (2011).
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they might encounter in the task. Similar methods might be used in the field to improve information acquisition among finance practitioners.24

The present finding that subjects acted like Bayesians in a complex decision-making task prompts the question of what makes a problem “complex” for the brain (“ecological relevance” may be the decisive criterion; see Brennan and Lo 2011) and more generally the relationship between problem difficulty and economic performance. Recent findings point to the intriguing possibility that economic performance actually increases with task difficulty, as people would expend more effort on more difficult problems, far beyond the point at which the marginal gain is positive (Murawski and Bossaerts 2016).

The current task is complex not only with regard to the amount of information to process during the task itself but also with regard to the amount of information that must be acquired during the task instructions. Providing the subjects with an exhaustive description of the stochastic structure of the task in the instructions may have “overloaded” them. Remarkably, it did not. This finding, together with prior evidence that people revert to adaptive learning if they are not provided with sufficient information about their environment (Payzan-LeNestour and Bossaerts 2015), suggests that the risk of overloading agents by disclosing too much information is less serious than the symmetrical pitfall—disclosing too little information. As such, the current findings speak to the complexity issue that has been much debated in the business world, concerning the right amount of information to disclose to financial agents.

While the evidence suggests that the subjects were overwhelmingly Bayesians in the present task, it should be noted that some localized departures from Bayesian behavior occurred. Specifically, a group of subjects behaved like the Bayesian model except after observing a black swan: they tended to bet thereafter, which constitutes a patent departure from both Bayesian and adaptive behaviors. The analysis of the behavior of those subjects is too lengthy to be included here; thus, it is the topic of a follow-up paper (Payzan-LeNestour 2018). Essentially, it appears that those subjects acted like “greedy” Bayesian learners: they learned about tail risk like in the Bayesian model but could not resist the temptation to bet after observing a black swan even though it was clear to them that the Bowman was an apprentice.

The present results raise a number of important questions. From the standpoint of practitioners, the finding that they need to follow the Bayesian approach if they are to survive under tail risk, leaves them with the question of whether regulators and firms could devise aids to Bayesian learning. The idea would be to “nudge” real-world agents into implementing the correct approach. A related direction for future research would consist of specifying the nature of the information needed for agents to be able to implement

24 By default, finance practitioners have access to a small number of independent observations of the DGP due to both observability and nonstationarity issues.
the Bayesian approach. The absence of structural uncertainty in the current experiment facilitated learning compared to the case in which some features of the stochastic structure of the task are disguised from the decision-maker. In the extreme case of full (“Knightian”) uncertainty, adaptive learning would surely prevail (possibly as an instance of “rational ignorance” inasmuch as Bayesian learning is too complex in that case). Where are the informational boundaries from which Bayesian learners switch to adaptive learning? Addressing that question is beyond the scope of behavioral studies because the absence of structural uncertainty is a pivotal aspect of removing the informational confound when using behavioral data, as explained above. However, the question may be addressed by using neural data, a method already used in similar contexts.25

Appendix A. Instructions for the Task

A.1 Original Experimental Sessions

The following pages show the text and static pictures used in the instructions for the original task. The instructions were displayed on Web pages and contained animations that the reader can find at http://bowmanexperiment.weebly.com/task-instructions.html. Immediately after the subjects went through the online instructions and completed the multiple-choice questionnaire, they were asked to carefully read the following FAQs sheet. The experimenter reviewed the FAQs orally with each subject before the subject began to perform the task.

25 See Martino et al. (2013), Payzan-LeNestour et al. (2013), and Frydman et al. (2014), among others.
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The following instructions explain the nature of the Bowman Game.

Please read them carefully.

The principle of the game is that on each trial, a Bowman is going to shoot an arrow towards a target on a wall (See Pic 1A: the wall is represented by the gray line; 0 represents the target).

![Picture 1A]

Before the Bowman shoots each arrow, you will have the opportunity to bet on whether the arrow will be close to the target or not. At each trial, before the Bowman shoots the arrow, your task will be to choose between two actions, Bet and Skip.

- By selecting *Bet* you will earn $2 if the arrow hits the wall up to four meters away from the target on the left or on the right (i.e., if the hit is anywhere within the segment [AB] on Pic. 1B), and you will lose $40 otherwise.

![Picture 1B]

- By selecting *Skip* you do not lose nor earn anything no matter where the arrow hits: you always get $0.

Once you have indicated your choice, the Bowman shoots and you see where the arrow hits. You’ll then move to the next trial, in which you’ll choose again between Bet and Skip, and then see the realized hit at that trial.

A play of the game is divided into 15 sessions of 20 trials each. For each session, a new Bowman is doing the shooting for the entirety of the session.
Each bowman is either a Master Bowman or an Apprentice Bowman, and has a specific shooting style corresponding to his skill level. We describe next the shooting style of the Master Bowman and then the Apprentice Bowman.

The Master Bowman

The hits of a Master Bowman are normally distributed around the target.

The dispersion of the hits from the target (the standard deviation) increases with the shooting distance of the bowman. The shooting distance differs across bowmen (see Pic. 2) but it is never changing within a session.

Picture 2

The standard deviation of the hits of a given bowman is unknown. It is uniformly distributed between 0.1 and 2 (i.e., all numbers within [0.1, 2] are equally likely).

The following interactive animation illustrates how the hits of the Master Bowman are distributed around the target, for different shooting distances (i.e., for different values of the standard deviation).

[show animation—distribution builders for Normal distributions here]

The Apprentice Bowman

The distribution of the hits of an Apprentice Bowman is bell shaped and symmetric around the target, like the one of the Master Bowman. But it's NOT NORMAL: with an Apprentice Bowman, you should not expect any particular hit to happen, because the hits can potentially be anywhere on the wall.
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Apprentice Bowman always shoot at the same distance from the target. The dispersion of their hits always equals 1, which implies that their hits are distributed around the target as follows:

[show animation—distribution builder for Cauchy distribution here]

Please note

Whether the Bowman for a session is a Master Bowman or an Apprentice Bowman is not told

Also note

The distance of the Bowman to the wall will vary from session to session, but the distance you’ll see on the screen will appear to be the same at each trial (again, this is NOT the real distance; the real distance varies from session to session).

This game is very hard, so you will need to be very concentrated to be successful. Your payment will be extremely sensitive to your performance during the game. Specifically, your payment will be the addition of:

- the fixed show-up reward of $5
- an account balance allocated to each participant at the start of the game (more on this below)
- all your accumulated earnings and losses during the entire game

The account balance is the same for all participants and has been fixed before the game begins. Its amount, which can be any number between 500 and 1500, will be revealed at the end of the game. This is because we don’t want you to be influenced by it during the game.

Your accumulated earnings and losses can be negative of course. In that case, the negative amount will be deducted from the money allocated to you at the start. If it goes below $5, you will still receive $5 as the show-up fee. If you are very good at the game, your final amount as calculated by the formula above might be more than $110. In that case, your payment will be capped at $110, as we simply can’t afford to pay each of you more than that. It may be challenging to reach this score though!

Good luck!

Just a quick reminder that on each trial during the game:

- by selecting Bet,
  You earn $2 if the arrow hits the wall up to four meters away from the target.
  You lose $40 if the arrow hits the wall beyond four meters away from the target.
- by selecting Skip,
  You do not lose nor earn anything no matter where the arrow hits. you always get $0.
One more thing: If you don’t answer within the imparted time (indicated by a timer), you lose $1, and you don’t see the hit realized at the trial, so try to keep the pace!

Please fill out the following multiple-choice questionnaire. This is to check your understanding of the rules of the game before you start to play. Thanks for your attention!

[multiple-choice questionnaire; for each question computer tells the subject whether answer is correct]

Multi-Choice Questionnaire

1. By playing Bet on a trial you earn $2 if the arrow shot by the bowman at the trial hits the wall up to four meters away from the target on the left or on the right, and you lose $40 otherwise (i.e., if the arrow hits the wall beyond four meters away).
   True/False

2. On each trial, the bowman doing the shooting can see, just before he shoots the arrow, whether you chose Bet.
   True/False

3. The sessions are completely independent: facing a Master Bowman in the current session does not make it more likely to face a Master Bowman on the following session. The chances to face a Master Bowman vs. an Apprentice Bowman are always 50-50.
   True/False

4. The dispersion (standard deviation) of the hits of a Master Bowman can be any number between 0.1 and 2. Before a session with a master begins, you are not told this number; it is fixed (will not change) throughout the session.
   True/False

5. The bowman doing the shooting will remain the same throughout a given session.
   True/False

6. The account balance that is allocated to you at the beginning of the game can be anything, any number between $500 to $500.
   True/False
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FAQS

1. Shall you expect the hits of a Master Bowman to reach the target on average?

YES! The distribution of the hits of any master Bowman is normally distributed around the target. It means that one expects the hits to reach the target on average; but of course this is not always the case for each individual hit. How far from the target the hits can be? The standard deviation tells you that:

- You saw in the simulations that if the std is 1 then the hits are all within -1 and +1
- If the std is 2 (the max that can happen) then there is a positive probability that the hits be outside [-2, +2]. But this probability is quite small

2. What does the standard deviation have to do with the distance at which the Bowman shoots?

You don’t care about the distance in itself. All you need to remember is that master Bowmen differ in the distance at which they do the shooting hence the standard deviations of their hits differ. It can be any number between 0.1 and 2.

It’s just that if the guy shoots from very far away the standard deviation is bigger than if the guy shoots from nearby the target.

3. Should we expect the std for a given master Bowman to be 0.1 or 2 or something else?

It is uniformly distributed between 0.1 and 2 meaning all values between 0.1 and 2 are equally likely.

4. What about the distribution of the hits of an apprentice Bowman? Is it normal too? The hits look pretty dispersed in the simulation so why is the dispersion only 1?

The distribution of the hits of an apprentice is NOT normal.

It’s bell-shaped and symmetric around the target, like the normal distribution, but it’s much more unpredictable than a normal. It is a very unpredictable (“fat tail”) distribution called Cauchy. Its dispersion parameter is 1. The simulation shows to you what it looks like exactly.

5. Are the hits of an apprentice Bowman and those of a master with a standard deviation of 2 pretty much the same?

NO they’re not!

The hits of the master Bowman with a std of 2 are most often within -4 and +4 and on expectation they reach at the target. By contrast, anything is possible with the hits of an apprentice! They’re completely unpredictable.

6. What does it mean that the sessions are independent an that the probability to face a master vs. an apprentice is 50-50?

What that means is that before each session, the computer tosses a coin to decide whether the Bowman in the next session is going to be a master or an apprentice. If it’s “head”, the Bowman will be an apprentice; if it’s “tail”, the Bowman will be a master.

The consequence of this is that you CANNOT infer from the nature of the Bowman in a given session whether the Bowman at the next session is going to be a master or an apprentice. In particular, it’s perfectly possible that you face an apprentice several sessions in a row (that’s bad luck!), or that you face a master several times in a row (lucky you if that happens!).

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A.2 Follow-up Experimental Sessions

The following shows the text of the instructions for the augmented version of the task (that in which the subjects were asked to guess the bowman’s type at the end of each session) as described in the main text. Highlighted in blue (but not highlighted in the instructions that the subjects read) are the parts that differ from the original instructions.

The following instructions explain the nature of the Bowman Game.

Please read them carefully.

The principle of the game is that on each trial, a bowman is going to shoot an arrow towards a target on a wall (See Pic 1A: the wall is represented by the gray line; 0 represents the target).

Picture 1A

Before the bowman shoots each arrow, you will have the opportunity to bet on whether the arrow will be close to the target or not. At each trial, before the bowman shoots the arrow, your task will be to choose between two actions, Bet and Skip:

- By selecting Bet you will earn $2 if the arrow hits the wall up to four meters away from the target on the left or on the right (i.e., if the hit is anywhere within the segment [AB] on Pic. 1B), and you will lose $40 otherwise.

Picture 1B

- By selecting Skip you do not lose nor earn anything no matter where the arrow hits: you always get $0.

Once you have indicated your choice, the bowman shoots and you see where the arrow hits. You’ll then move to the next trial, in which you’ll choose again between Bet and Skip, and then see the realized hit at that trial.

A play of the game is divided into 15 sessions of 20 trials each. For each session, a new bowman is doing the shooting for the entirety of the session.
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Each Bowman is either a Master Bowman or an Apprentice Bowman, and has a specific shooting style corresponding to its skill level. We describe next the shooting style of the Master Bowman and then the Apprentice Bowman.

The Master Bowman

The hits of a Master Bowman are normally distributed around the target.

The dispersion of the hits from the target (the standard deviation) increases with the shooting distance of the bowman. The shooting distance differs across bowmen (see Pic. 2) but it is never changing within a session.

![Picture 2]

The standard deviation of the hits of a given bowman is unknown. It is uniformly distributed between 0.1 and 2 (i.e., all numbers within [0.1; 2] are equally likely).

The following interactive animation illustrates how the hits of the Master Bowman are distributed around the target, for different shooting distances (i.e., for different values of the standard deviation).

[show animation—distribution builders for Normal distributions here]

The Apprentice Bowman

The distribution of the hits of an Apprentice Bowman is bell shaped and symmetric around the target, like the one of the Master Bowman. But it’s NOT NORMAL: with an Apprentice Bowman, you should not expect any particular hit to happen, because the hits can potentially be anywhere on the wall.
Apprentice Bowman always shoot at the same distance from the target. The dispersion of their hits always equals 1, which implies that their hits are distributed around the target as follows:

[show animation—distribution builder for Cauchy distribution here]

Please note

Whether the bowman for a session is a Master Bowman or an Apprentice Bowman is not told.

Also note

The distance of the bowman to the wall will vary from session to session, but the distance you’ll see on the screen will appear to be the same at each trial (again, this is NOT the real distance; the real distance varies from session to session).

This game is very hard, so you will need to be very concentrated to be successful. Your payment will be extremely sensitive to your performance during the game. Specifically, your payment will be the addition of:
- the fixed show-up reward of $5
- an account balance allocated to each participant at the start of the game (more on this below)
- all your accumulated earnings and losses during the entire game

The account balance is the same for all participants and has been fixed before the game begins. Its amount, which can be any number between -500 and +500, will be revealed at the end of the game only. This is because we don’t want you to be influenced by it during the game.

Your accumulated earnings and losses can be negative of course. In that case, the negative amount will be deducted from the money allocated to you at the start. If it goes below $5, you will still receive $5 as the show-up fee. If you are very good at the game, your final amount as calculated by the formula above might be more than $110. In that case, your payment will be capped at $110, as we simply can’t afford to pay each of you more than that. It may be challenging to reach this score though!

Good luck!

One more thing

At the end of each session, you will be asked your opinion about the bowman you’ve faced in the session (whether he was a master or an apprentice).

A correct answer yields $10. An incorrect answer results in a loss of $10.

If you don’t answer within the imparted time (20 sec), you’ll lose $12.
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Just a quick reminder that on each trial during the game:

- by selecting Bet,

You earn $2 if the arrow hits the wall up to four meters away from the target. You lose $40 if the arrow hits the wall beyond four meters away from the target.

- by selecting Skip,

You do not lose nor earn anything no matter where the arrow hits: you always get $0.

One more thing: If you don’t answer within the imparted time (indicated by a timer), you lose $1, and you don’t see the hit realized at the trial, so try to keep the pace!

Please fill out the following multiple-choice questionnaire. This is to check your understanding of the rules of the game before you start to play. Thanks for your attention!

[multiple-choice questionnaire; for each question computer tells the subject whether answer is correct]

Multiple-Choice Questionnaire

1. By playing Bet on a trial you earn $2 if the arrow shot by the Bowman at the trial hits the wall up to four meters away from the target on the left or on the right, and you lose $40 otherwise (i.e., if the arrow hits the wall beyond four meters away).

   True/False

2. On each trial, the Bowman doing the shooting can see, just before he shoots the arrow, whether you chose Bet.

   True/False

3. The sessions are completely independent; facing a Master Bowman in the current session does not make it more likely to face a Master Bowman on the following session. The chances to face a Master Bowman vs. an Apprentice Bowman are always 50-50.

   True/False

4. The dispersion (standard deviation) of the hits of a Master Bowman can be any number between 0.1 and 2. Before a session with a master begins, you are not told this number; it is fixed (will not change) throughout the session.

   True/False
5. The bowman doing the shooting will remain the same throughout a given session.

True/False

6. The account balance that is allocated to you at the beginning of the game can be anything, any number between $-500 to $500."

True/False

7. At the end of each session, you will be asked to give your guess about the nature of the bowman in the session. If you reply correctly you win $10, but if you reply incorrectly you lose $10. If you fail to reply, you lose $12.

True/False

Appendix B. Behavioral Models

B.1 Bayesian Learning

B.1.1 Inferring the DGP. At each trial \( t \) of a given session, the Bayesian agent compares the plausibility (likelihood) of the two possible models of the world (Normal versus Cauchy) given the data available until (and including) trial \( t \):

- Under the Normal DGP \((M_1)\) (i.e., if the bowman is a master), the data \( X_t = (X_1, X_2, \ldots, X_t) \):

\[
f_1(X_t | \sigma) = \prod_{k=1}^{t} \frac{1}{\sqrt{2\pi \sigma}} \exp\left\{ -\frac{X_k^2}{2\sigma^2} \right\}, \quad (B1)
\]

where \( \sigma \) (the unknown standard deviation) is uniformly distributed between 0.1 and 2.

- Under the Cauchy DGP \((M_2)\) (i.e., if the bowman is an apprentice), the data \( X_t \):

\[
f_2(X_t) = \prod_{k=1}^{t} \frac{1}{\pi (X_k^2 + 1)} , \quad (B2)
\]

At each trial, the Bayesian model assesses how likely it is that the session is Normal versus Cauchy in light of the available data \( X_t \). The metric used is the “marginal or predictive density” (Berger and Pericchi 2001) of \( X_t \) under each model \((M_1 \) and \( M_2)\):

\[
m_1(X_t) = \int_{0.1}^{2} f_1(X_t | \sigma) \times \pi(\sigma) \, d\sigma , \quad (B3)
\]

\[
m_2(X_t) = f_2(X_t) , \quad (B4)
\]

where \( \pi(\sigma) \) denotes the prior used for \( \sigma \). The uniform prior \( \pi(\sigma) = \frac{1}{\sigma} \) may appear to be a natural choice, although the “reference” (noninformative) prior \( \pi(\sigma) = \frac{1}{\sigma} \) actually may be preferred according to prior statistics studies which commonly advocate its use for model selection (e.g., Berger and Pericchi 1996; Betolino and Racugno 1997). Both priors were tested in simulations, and the behavior of the model does not appear to be sensitive to the choice of the prior.

The evidence for the Normal DGP \((M_1)\) against the Cauchy DGP \((M_2)\) is provided by the Bayes factor of \( M_1 \) to \( M_2 \):

\[
B_{12}(t) = \frac{m_1(X_t)}{m_2(X_t)} . \quad (B5)
\]

For instance, if \( B_{12} = 3 \), the Normal hypothesis is favored over the Cauchy hypothesis at odds of 3 to 1. To compute the expected value of each action, I use the Bayesian model to first assess the
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posterior probability of each model given the data $X_t$, which is a simple transform of $B_{12}$, as the prior model probabilities, denoted $P(M_1)$ and $P(M_2)$, equal to 1/2:

$$P(M_1 | X_t) = \frac{P(M_1) m_1(X_t)}{P(M_1) m_1(X_t) + P(M_2) m_2(X_t)} = \frac{1}{1 + B_{12}}, \quad (B6)$$

$$P(M_2 | X_t) = \frac{P(M_2) m_2(X_t)}{P(M_1) m_1(X_t) + P(M_2) m_2(X_t)} = \frac{1}{B_{12} + 1}, \quad (B7)$$

**B.1.2 Decision.** From these posterior probabilities, the Bayesian model directly derives the expected value of action bet at trial $t$, which is denoted $V_t$:

$$V_t = P(M_1 | X_{t-1}) \left[ p_x \times 2 - (1 - p_x) \times 40 \right] + P(M_2 | X_{t-1}) \left[ p_x \times 2 - (1 - p_x) \times 40 \right], \quad (B8)$$

where $p_x$ (resp. $p_y$) denotes the probability of a winning bet ($X_t \in [-4,4]$) in a session with a master bowman (resp. with an apprentice bowman).

$p_x$ is computed for each trial $t$ as a function of the standard deviation estimate $\tilde{\sigma}(t) = \sqrt{\frac{1}{t} \sum_{t=1}^{t} X_t^2}$. The Bayesian agent uses this estimate to assess the likelihood of a winning bet in a session with a master bowman to be

$$p_x = p_x(t) = 1 - 2 \left(1 - \Phi \left( \frac{4}{\tilde{\sigma}(t)} \right) \right). \quad (B9)$$

The probability of a winning bet ($X_t \in [-4,4]$) in a session with an apprentice bowman is computed as follows (using the definition of the Cauchy density):

$$p_y = \frac{1}{\pi} \int_{-4}^{4} \frac{1}{x^2 + 1} dx = \frac{1}{\pi} \left[ \tan^{-1}(4) - \tan^{-1}(-4) \right] = 0.844. \quad (B10)$$

Because action skip always yields 0 regardless of the state of the world, the optimal decision simply consists of betting if the metric $V$ (the expected value of action bet, as defined in Equation (8)) is positive and skipping otherwise. Bayesian behavior thus consists of betting only when/if $B_{12}$, the evidence that the bowman is a master, becomes sufficiently large.

Note that the foregoing assumes that the agent is risk neutral. The behavior of the Bayesian agent may alternatively reflect loss aversion and the distortion of the probabilities, like in prospect theory (Kahneman and Tversky 1979). The expected value of action bet for the prospect theory Bayesian agent is

$$V_t = P(M_1 | X_{t-1}) \left[ w(p_x) \times 2^{\alpha_1} - \lambda \times w(1 - p_x) \times 40^{\alpha_2} \right] + P(M_2 | X_{t-1}) \left[ w(p_x) \times 2^{\alpha_1} - \lambda \times w(1 - p_x) \times 40^{\alpha_2} \right], \quad (B11)$$

where $\alpha_1$ and $\alpha_2$ govern the shape of the value function, $\lambda$ is the loss-aversion parameter, and $w(.)$ is the probability-weighting function introduced by Prelec (1998): $w(p) = \exp\left[-(\ln p)^{\alpha_3}\right]$. $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\lambda$ are free parameters in the model comparison analysis.

It is also possible that the subjects felt disappointed after a loss and accounted for their disappointment ex ante in their valuation of action bet. The derivation of the $V$ metric in that case is standard but somewhat lengthy (available on request). The behavior of the loss-averse/disappointment-averse agent is essentially the same as that of the risk-neutral model.

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26 Recall that at the onset of each session, the agent knows that there is an equal chance of being in a session with a master and being in a session with an apprentice.
only difference is that compared to the case in which the agent is risk neutral, the agent begins betting later in the sessions with a master, as the value of $B_{12}$ from which the agent begins betting is higher.

Finally, the suboptimal Bayesian variant described in the main text learns about the world model like the Bayesian model does, but it uses a “classical” decision rule in place of Bayesian model averaging (see Equation (8) above); that is, the agent places full weight on the hypothesis deemed more likely. The expected value of action bet at trial $t$ is thus

$$V_t = \begin{cases} p_B \times 2 - (1 - p_B) \times 40 & \text{if } P(M_1 | X_{t-1}) > P(M_2 | X_{t-1}), \\ p_S \times 2 - (1 - p_S) \times 40 & \text{otherwise.} \end{cases}$$

**B.2 Adaptive Learning**

**B.2.1 EWA model.** The value or “attraction” of each action $i \in \{B, S\}$ (where $B$ and $S$ represent “bet” and “skip,” respectively), denoted by $A_i$, depends on the past payoff experienced with the two actions. Specifically, let $\pi_i(t)$ denote the payoff of action $i$ at trial $t$ of a given session:

$$\begin{aligned} \pi_B(t) &= \begin{cases} 2 & \text{if } X_t \in [-4, 4], \\ -40 & \text{otherwise.} \end{cases} \\
\pi_S(t) &= 0. 
\end{aligned}$$

(B12)

Within each session, the attractions are updated at each trial $t$ as follows:

$$A_i(t) = \frac{\phi N(t-1) A_i(t-1) + \delta + (1 - \delta) I(i(t))}{N(t)} \pi_i(t),$$

(B14)

where

$$I(i(t)) = \begin{cases} 1 & \text{if } i \text{ is chosen at trial } t \\ 0 & \text{if } i \text{ is not chosen at trial } t. \end{cases}$$

Under this approach, the behavior consists of choosing action bet in trial $t+1$ if and only if $A_B(t) > A_S(t)$ and choosing action skip otherwise. $N(t)$ measures the amount of experience that the subject has accumulated after trial $t$ has taken place. $N(t)$ is updated as follows (starting with an initial value $N(0)$):

$$N(t) = (1 - \kappa) N(t-1) + 1.$$

(B15)

The free parameters in the analysis are $N(0), \phi, \kappa,$ and $\delta$, which determine the learning rate (i.e., the weight placed on prior beliefs compared to the payoffs), how quickly previous experience is discarded, the growth rate of attractions, and the relative weight given to the actual payoff compared to the forgone payoff, respectively. When $\delta = 0$, the agent entirely ignores forgone payoffs; when $\delta = 1$, the agent weights the forgone payoff as heavily as the actual payoff. In principle $A(0)$, the initial value of $A(t) = (A_B(t), A_S(t))$ should be approximately $(-1.57, 0)$ to reflect the expected value of the two actions, but the main conclusions are robust to allowing the parameter to depart from the expected value and treating it as a free parameter. When $\kappa = 0$, $\delta = 1$, and $A(0)$ equals the prior expected values of the two actions, the EWA attractions are the same as the expected payoffs according to “weighted fictitious play,” which, in many contexts, is indistinguishable from

$$27 \text{ The prior value of action } S, A_S(0), \text{ is zero because action } S \text{ yields } 0. \text{ Regardless of } A_B(0), \text{ the agent knows that there are equal chances that the bowman is a master versus an apprentice and that the chance of a winning bet is, on average, approximately } 99\% \text{ if the bowman is a master and } 84\% \text{ if the bowman is an apprentice. The prior expected value of action } B \text{ is therefore } \frac{1}{2} \left( 0.84 \times 2 \times (1 - 0.84) \times (-40) \right) + \frac{1}{2} \left( 0.99 \times 2 \times (1 - 0.99) \times (-40) \right) = -1.57.$$

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Figure B1
How the parameters of the EWA model can be fine-tuned to match a wide range of behaviors in the task.
In this simulation with a master bowman (normal DGP), the standard deviation of the simulated shots was 1.5. All of the simulated shots fell within [−4;4], except for the shot in trial 17. In the simulation reported in the top figure, the specification of the EWA model was as follows: \( \phi = 1, \kappa = 0, A_B(0) = -6, \) and \( N(0) = 1. \) In the middle figure, the specification of the EWA was the same, except that \( N(0) = 0.5 \) (i.e., the agent learned quickly). In the bottom figure, the specification of the EWA was as in the top graph except that \( \delta = 0.5 \) (i.e., the agent hybridized features of counterfactual and reinforcement learning).

Legend: Each figure shows the value of action bet \( (A_B) \) and skip \( (A_S) \) (top), the corresponding choice made by the agent (bottom) and the ensuing accumulated outcomes (middle).

Bayesian learning (Camerer and Ho 1999). Notably, this is not the case in the current task, as reported in Section 2.2.

To intuitively show the effect of each parameter and how the parameters can combine to generate the observed behavior, Figure B1 reports the behaviors of three simulated EWA models in a session with a master bowman (i.e., the DGP was normal) in which all of the shots fell within [−4;4], except for the shot in trial 17. The value of the attractions and the corresponding choice and the ensuing outcome are indicated for each trial of the session. In the simulation reported in the top graph, the specification of the EWA model was as follows: \( \phi = 1, \kappa = 0, A_B(0) = -6, \) and \( N(0) = 1. \) In the middle graph, the specification of the EWA model was the same as in the top graph except for
learning, meaning that it took longer for the model to build the attraction value learning (the curve for $A_{\delta}$ is steeper in the middle graph than in the top graph). As a result, the model began betting from the third trial, whereas it did not bet until the fifth trial when $N(0)=1$.

In the bottom graph, the specification of the EWA model was like that in the top graph, except for $\delta$, which was reduced by half. The effect of decreasing $\delta$ is to reduce the extent of counterfactual learning, meaning that it took longer for the model to build the attraction value $A_{\delta}$ in the early stage of the session. As a result, the model did not bet until the ninth trial. A similar method used to successively vary the value of $\kappa$, $A_{\delta}(0)$, and $\phi$, reveals the following:

1. EWA behavior does not appear to be sensitive to the value of $\kappa$ in the present task as long as $\phi$ and $A_{\delta}(0)$ are not too low.$^{28}$

2. decreasing $A_{\delta}(0)$ is effective in making the EWA model bet later in a given session;

3. decreasing the value of $\phi$ (i.e., increasing forgetting) is effective in making the EWA model bet after/despite experiencing a loss as long as $A_{\delta}(0)$ is close to the aforementioned benchmark value of $-1.57$. However, if $A_{\delta}(0)$ markedly departs from the benchmark to take on a very negative value (which is necessary to make the EWA model skip in the first trials of the session), there is no combination of the parameters that can make the model bet after experiencing a loss.

The foregoing thus shows that there is no combination of the parameters that can make the EWA agent behave like the Bayesian model (which both skips early in each session and perseveres in betting after experiencing a loss if the shot that led to that loss was a near miss, as explained in the main text).

B.2.2 Adaptive forecaster model. The adaptive forecaster model is an adapted version of the algorithm originally proposed by Foster and Vohra (1998) and revisited by Carvajal (2009). In each trial, the model purports to forecast in a purely adaptive way—that is, without inferring the DGP—with/without experiencing a loss if the shot that led to that loss was a near miss, as explained in the main text).

$$\phi_m = \frac{1}{M} \sum_{k=1}^M I(p_k = p(m)),$$  \hspace{1cm} (B17)

where $I(p_k = p(m))$ equals 1 if $p_k = p(m)$ and 0 otherwise. The model further considers “the excess” associated with the candidate $p(m)$:

$$e^p_m(h_t) = \left(\frac{1}{M} \sum_{k=1}^M I(p_k = p(m)) \bigg/ t\right),$$  \hspace{1cm} (B18)

28 When $\kappa = 0$ and $A_{\delta}(0)$ are very small, the EWA agent commits to skipping in all sessions.
and the following metric (called "deficit")

\[ d_t^n(h_t) = \left( \frac{m-1}{M} - p_t^n(h_t) \right) \sum_{k=1}^{t} I_{p_t^n \neq p_{(k-1)}^n}. \]  \hfill (B19)

The initial value of the payoff probability estimate \( p_1 \) may take on different values. It may seem natural to set it to the candidate probability estimate closest to the true prior probability of a winning bet, which is approximately 0.91 by design.\(^{29}\) That is, \( p_1 = p(M) \) for \( M \leq 11 \) and \( p_1 = p(M - 1) \) otherwise (if 11 < \( M \leq 22 \)).\(^{30}\) In principle, the task participants had a number close to that value in mind given the information provided in the task instructions. I nonetheless considered other possibilities. In particular, I tested a variant of the model in which the agent is initially agnostic about payoff probability and hence randomly chooses one probability estimate from the set of candidates. I also tested the semiangnostic version of the model, which randomly chooses one probability estimate from the restricted set \{ \( p(M/2), \ldots, p(M) \) \}. One may find the semiangnostic version more realistic than the other inasmuch as the idea that subjects were entirely ignorant of the payoff probability, despite the transparency of the task instructions, is arguably not tenable (but I nevertheless accounted for both possibilities). In all of the results reported below, the optimal version of the model was used because it performed markedly better than both agnostic variants, which are therefore not discussed further in this paper. The conclusions of the model comparison between the adaptive forecaster and the other models were exactly the same whatever the method used to define \( p_1 \).

For \( t \geq 2 \), the rule for choosing \( p_1 \) is that if there exists some candidate \( \tilde{m} \in M \) such that \( p_t^n(\tilde{m}) \in \{ \tilde{m}(\tilde{m}) \} \), then \( L_t(h_{t-1})(\tilde{m})=1 \) and \( L_t(h_{t-1})(m)=0 \) for every other \( m \). Otherwise, the model searches for the candidate \( \tilde{m} \) such that \( p_t^n(\tilde{m}) > 0 \) and \( e_t^{n-1}(h_{t-1}) > 0 \) and sets the probability of choosing each candidate \( m \) as follows:\(^{31}\)

\[ L_t(h_{t-1})(\tilde{m}) = \frac{e_t^{n-1}(h_{t-1})}{d_t^{n-1}(h_{t-1}) + e_t^{n-1}(h_{t-1})}. \]  \hfill (B20)

\[ L_t(h_{t-1})(\tilde{m}) = 1 - L_t(h_{t-1})(\tilde{m}), \]  \hfill (B21)

\[ L_t(h_{t-1})(m) = 0 \quad \text{for all other } m. \]  \hfill (B22)

Then, to determine \( p_t \), the model selects one candidate from the set \{ \( p(1); p(2), \ldots; p(M) \) \} according to the stochastic rule \( \{ L_t(h_{t-1})(m) \}_{M_{\tilde{m}}+1}^{M} \).\(^{32}\) Thus, the adaptive forecaster model learns the payoff probability in a purely adaptive fashion in that it is completely ignorant of both the real probability of the occurrence of the event \( (Z_t = 1) \) as well as the reasons that the event occurs or fails to occur. Moreover, the model does not learn anything about the future from the history.

Once \( p_t \) is determined, the expected value of action bet at trial \( t \) is computed in the usual way:

\[ V_t = p_t \times 2 - (1 - p_t) \times 40. \]  \hfill (B23)

\(^{29}\) The prior probability of a winning bet is approximately 0.5 x 0.99 + 0.5 x 0.84 = 0.91 because, by design, there is an equal chance of facing a master and an apprentice, and the probability of a winning bet is approximately 0.99 on average with a master and 0.84 with an apprentice.

\(^{30}\) The sensible range of values for \( M \) is between 9 and 22. Outside this range, behavior appeared to be degenerate in simulations (e.g., betting in all trials).

\(^{31}\) If there is no such \( \tilde{m} \), the model determines \( p_t \) by randomly picking one candidate \( m \) among the hitherto unchosen probability candidates.

\(^{32}\) That is, the first candidate, \( p(1) \), is chosen with probability \( L_t(h_{t-1})(1) \); the second, \( p(2) \), is chosen with probability \( L_t(h_{t-1})(2) \), etc.
Figure C1
Optimization of the loss-averse Bayesian model.

The objective function to be minimized is the mean squared prediction error (y-axis). Here, the model that is estimated is the loss-averse Bayesian model with three free parameters: the “value function” parameters $\alpha_1$ and $\alpha_2$ and the “loss aversion” parameter $\lambda$, as defined in the main text. For each of these parameters, the objective function is plotted as a function of the parameter value (x-axis) at the optimum value of the other free parameters. The function is concave in the $\alpha_1$, $\alpha_2$, and $\lambda$ dimensions, allowing for minimization and the successful estimation of the model.

Appendix C. Optimization Procedure

I fitted the Bayesian and EWA models to each task participant’s choices and then compared the goodness-of-fit of the best-fit Bayesian and best-fit EWA models. Specifically, the procedure proceeded in two steps, a first “in-sample” calibration (estimation) step followed by an “out-of-sample” validation test. In the first step, the free parameters of each model were chosen separately for each subject to maximize the proximity of the model predictions to the actual choices in the first eight sessions. The parameters were optimized by minimizing the total squared prediction error compounded over the set of trials (see Figure C1). The ensuing parameter estimates were then used in a second stage of the analysis to predict the choices in the last seven sessions (out-of-sample validation). The success rate of each model was measured by the percentage of trials in which the model successfully predicted the subject’s choice in the last seven sessions. For each subject, the out-of-sample success rate of the best-fit EWA model was compared to that of the best-fit Bayesian model—loss-averse or disappointment averse—and, for reference, to that of the risk-neutral version of the Bayesian model for which there was no free parameter (i.e., no in-sample calibration was needed).

This type of two-step procedure is commonly used in model comparison analyses involving very general models, such as the EWA model, to guard against overfitting. Very general models often fit well in-sample by overfitting, and modifying the standard goodness-of-fit criteria to account for differing degrees of freedom across models (cf. Bayesian information criterion, Akaike information criterion, and the like) is not sufficient; forecasting out-of-sample is critical to limit the advantage of general models over simpler contenders. See, for example, Camerer and Ho (1999) and Ho et al. (2007, 2008).33

Initially, I implemented a grid search to find the best values for the parameters.34 Next, these best values were used as the initial values for the constrained nonlinear optimization algorithm patternsearch implemented in Matlab (The MathWorks, Inc.), which further refined the search for
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the optimum. I checked that the goodness-of-fit results are robust across different optimization methods to discover the optimum.

As for the adaptive forecaster model, the foregoing method could not be used to fit its free parameters because the model is stochastic by nature. Thus, I assessed the success rate of the adaptive forecaster model for different starting conditions and for different fixed values of \( M \) in the sensible range (which is between 9 and 22, as explained above). The goodness-of-fit of the adaptive forecaster model appeared to be fairly robust across the different values of \( M \) in the sensible range. The best fit was obtained for \( M =15 \). The results of the model comparison analysis reported in Section 2.1.3 were those obtained using the best-fit version.

Importantly, all the results documented below remained consistent through an extensive set of robustness checks, which included fitting the models using maximum likelihood (subject choice was modeled using the logit rule) rather than least squares techniques, using different optimization methods to discover the optimum in the in-sample calibration phase, and using different lengths for the out-of-sample validation test.

References


By “stochastic,” I mean here that when facing the exact same data twice sequentially, the model may choose a different probability candidate, leading to a different choice in the two instances with the same data.


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